Single Particle Level Density in a Harmonic – Oscillator Potential Well

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Abstract

The purpose of this paper is to study the properties of the partial level density $g_l(\varepsilon)$ and the total level density g (ε), numerically obtained as a l sum of $g_l(\varepsilon)$ up to $l_{\max} \leq 34$, for a Harmonic – Oscillator potential well. This method applied the quantum – mechanical phase shift technique and concentrated on the continuum region. Also a discussion of peculiarities of quantal calculation for single particle level density of energy – dependent potential.

Keywords
Level Density
Harmonic – Oscillator
Potential Well

Article info
Received: Mar. 2010
Accepted: Apr. 2010
Published: Jan. 2011

كثافة مستويات الجسيمة المنفردة في بئر الجهد المتذبذب التوافقي

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الخلاصة

الغرض من هذا البحث هو لدراسة خواص كثافة مستويات الطاقة الجزئية و كثافة مستويات الطاقة الكلية و التي تحسب عددياً كحاصل جمع الزخم الزاوي لكثافة مستويات الطاقة الجزئية إلى أعلى قيمة للتحليل ألموجي و باستخدام الجهد ألبئري التوافقي ، وحيث يتم استخدام تقنية فرق الطور الكمية ، ويتم التركيز على المنطقة المتصلة (أي خارج منطقة الجهد) . كذالك يتم مناقشة بعض الحسابات الكمية الخاصة لحساب كثافة مستويات الطاقة للجسيم المنفرد .

Introduction

An essential element of the theory of nuclear structure and nuclear reactions is the single particle level density $g(\varepsilon)$, associated with nucleus mean field.

The nucleus level density ρ(ε) needed for the description of nuclear reactions, also the quasiparticle level density used in describing pre – equilibrium nuclear reaction , and for the calculation of the partition function.

We stress that for an accurate description of these physical quantities one needs to know $g(\varepsilon)$ for a wide range of (E), including the continuum region.

The method used in the literature [1 - 5] for descretizing the continuum, using adopting an infinite potential well, by diagonalsing the finite well Hamiltonian in the space associated with an infinite well Hamiltonian, when the nucleus putting in an infinite box or by locating the energies of the single particle resonances . This leads to a single particle level density $g(\varepsilon)$which increases with (ε) in the continuum.

A proper accounting for the continuum is important for determining nuclear properties, especially of an excited nucleus [5].

Formalism

We consider a particle, such as the nucleon, moving in a single particle mean field and follow the effective potential.

$$V(r) = \frac{1}{2}\, m\omega^2 r^2 + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$$

where m = the mass

$\omega = 2\pi\,\nu$

$\hbar = h/2\pi$ *Blanck's Constant*

l = *Angular Momntum*

For a given single particle Hamiltonian is

$$\hat{H} = \frac{p^2}{2m} + V(r) \quad \text{-----------------(1)}$$

For an infinite potential, V(r) , all eigen states of ($\hat{H}$) are bound and g(ε) is given by .

$$g(\varepsilon) = \sum_i \delta(\varepsilon - \varepsilon_i) \quad \text{----------(2)}$$

where ε_i are eigenenergies obtained from .

$$\hat{H}\psi_i = \varepsilon_i \psi_i \quad \text{--------------------(3)}$$

and ψ_i is the given Eigen function for the single particle .

Our main objective is to consider the case of a finite Harmonic-Oscillator potential , such as the mean potential describing a nucleon in the nucleus . In this case the incident particle spectra consists of bound states at $\varepsilon \le 0$, and unbound continuum states at $\varepsilon > 0$. We will concentrate in this work on the study of the continuum states . If we contain the system described by eq.(1) in an infinite spherical box with a radius R larger than the range of V(r) , the continuum would be discretized . However , the level density $g(\varepsilon)$calculated from eq.s(2) &(3) , will depend mainly on R . As it increases with R for $\varepsilon > 0$. This is due to the fact that $g(\varepsilon)$ includes the contribution of the so – called free – gas level density $g_f(\varepsilon)$ which can be determined by using the free particle Hamiltonian:-

$$\hat{H}_0 = \frac{p^2}{2m} \quad \text{---------------------(4)}$$

Therefore, the single particle level density associated with a finite potential, V(r), can be obtained from.

$$g(\varepsilon) = \lim_{R\to\infty}\left(g_v(\varepsilon) - g_f(\varepsilon)\right) \text{---(5)}$$

where g_v *and* g_f are calculated by using $\hat{H}$ & $\hat{H}_0$, respectively . In this work we consider the single particle level density for a neutron in a nucleus , using a Harmonic – Oscillator potential well without a spin – orbit term, carrying out an angular momentum decomposition of g we write :-

$$g(\varepsilon) = \sum_{l=0}^{l_{max}} 2(2l+1) g_l(\varepsilon) \quad \text{----(6)}$$

where the factor $2(2l+1)$ represent the counting of spin and the spherical potential degeneracy, and $g_l(\varepsilon)$, represent the partial level density which can be obtained by two contributions, one of bound states contribution , that labeled $g_{Bl}(\varepsilon)$, and the second one is labeled $g_{cl}(\varepsilon)$, which represents the contribution of the states in the continuum, then

$$g_l(\varepsilon) = g_{Bl}(\varepsilon) + g_{cl}(\varepsilon) \text{----(7)}$$

where $g_{Bl}(\varepsilon)$ is the contribution of the bound states of $\hat{H}$ with Eigen energies (ε_{nl}) , and it is given by .

$$g_{Bl}(\varepsilon) = \sum_{\varepsilon_{nl\le 0}} \delta(\varepsilon - \varepsilon_i) \text{----(8)}$$

Determination the contribution of states in the continuum $g_{cl}(\varepsilon)$

In order to find the single particle level density in the continuum region, we should firstly find the Eigen function that combined the neutron in the continuum. The undisturbed wave is represented by the wave plane $e^{ik.z}$ (Born approximation), and $e^{ik.z}$, can be represented by the spherical Harmonic wave function as follows:

$e^{ik.z} = \sum_l A_l(r)\, Y_{l,0}(\theta)$ -------(9),

then,

$$A_l(r) = \frac{\sqrt{\pi}}{kr}\sqrt{2l+1}\, i^{l+1}\left[e^{-i(kr-l\pi/2)} - e^{i(kr-l\pi/2)}\right] Y_{l,0}(\theta)$$

--------------------(10)

The first term between the brackets represent, $\psi_{incoming}$, and the second term represent , $\psi_{outgoing}$, while the scatterer affect only on the outgoing wave, so let this effect is represented by the factor , (η) , then ,

$$\psi(r) = \frac{\sqrt{\pi}}{kr}\sum_{l=0}^{\infty}\sqrt{2l+1}\, i^{l+1}\left[e^{-i\left(kr-\frac{l\pi}{2}\right)} - \eta e^{i\left(kr-\frac{l\pi}{2}\right)}\right] Y_{l,0}(\theta)$$

----------------(11)

so the outgoing wave is

$$\psi(r) = \frac{\sqrt{\pi}}{kr}\sum_{l=0}^{\infty}\sqrt{2l+1}\, \eta\, e^{i\left(kr-\frac{l\pi}{2}\right)}\, Y_{l,0}(\theta)$$

----------------(12)

Now let $\eta = e^{i\Delta_l(\varepsilon)}$, so,

$$\psi(r) \sim cost.\frac{1}{r}\; e^{i\left(kr-\frac{l\pi}{2}+\Delta_l(\varepsilon)\right)}$$

-----------------(13)

From the boundary condition that, $\psi_l(0) = 0$, then

$$\psi(r) \sim cons.\frac{1}{r}\sin\left[kr - \frac{l\pi}{2} + \Delta_l(\varepsilon)\right]$$

-----------------(14)

which is asymptotic to the solution of the wave equation in classical mechanic ,

$$x = A\sin(\omega t)$$

now since , $\psi_l(R) = 0$, then equation (14) equal to zero . For that ,

$kR - \frac{l\pi}{2} + \Delta_l(\varepsilon) = s\pi$ --------(15)

where,

$s \equiv is\ integer\ number\ 1,2,3, \ldots..$

Then by taking the derivative with respect to (ε), and multiplying by $2(2l+1)$, we obtain

$$g_{cl}(\varepsilon) = \frac{2(2l+1)}{\pi}\frac{d\Delta_l(\varepsilon)}{d(\varepsilon)} + \frac{2(2l+1)}{\pi}\frac{dk}{d\varepsilon}$$

--------------(16)

It is easily to recognized that the second term of eq.(16) is proportional directly to (R), where its contribution of free gas obtained from, (H_0), using a spherical box of radius (R) , and if we subtract the contribution of the free gas, we obtain :-

$g_{cl}(\varepsilon) = \frac{2(2l+1)}{\pi}\frac{d\Delta_l(\varepsilon)}{d\varepsilon}$ ---------(17)

Finally the total partial level density, $g_l(\varepsilon)$ can be written as.

$$g_l(\varepsilon) = \sum_{\varepsilon_{nl\le 0}} 2(2l+1)\,\delta(\varepsilon-\varepsilon_{nl}) + \frac{2(2l+1)}{\pi}\frac{d\Delta_l(\varepsilon)}{d\varepsilon}$$

---------------------(18)

Determination of phase shift

The relation govern the phase shift is given by [6].

$\tan\Delta_l = \frac{kR\, j_l'(kR) - B_l j_l(kR)}{kR n_l'(kR) - B_l n_l(kR)}$----(19)

Where $k = \sqrt{\frac{2mE}{\hbar^2}}$ is the wave number and
$R = being\ the\ range\ of\ potential$

$j_l(kR) \equiv Spherical\ Bessel\ Function$
and
$n_l(kR) \equiv Spherical\ Neumann\ Function$

It is common to use the logarithmic derivative, $B_l = \left(\frac{r}{A_l(r)}\frac{dA_l(r)}{dr}\right)_{r=R}$, to determine Δ_l, and investigate its properties . This computed for the wave function at distance, r=R ,

$$A_l(r) = e^{i\Delta_l}[\cos\Delta_l \ \ j_l(kR) - \sin\Delta_l \ \ n_l(kR)] \quad \text{---------(20)}$$

and, $u_l = r\,A_l(r)$ ------------------(21)

As shown from eq.(19) , that the problem of determining the phase shift is just how to obtain , B_l .

Now, B_l , can be found by solving Schrödinger equation for , $r < R$, (that is inside the range of potential). For a spherical symmetric potential, we can solve Schrödinger equation in three dimensions by looking at the equivalent one – dimensional equation.

$$\frac{d^2u_l}{dr^2} + \left(k^2 - \frac{2m}{\hbar^2}V - \frac{l(l+1)}{r^2}\right)u_l = 0 \quad \text{------------------(22)}$$

Whereas it was mentioned previously in , eq.(21) , then

$$A_l(r) = \frac{U_l}{r} \quad \text{-----------------(23)}$$

Also we have adopted for the spherical Bessel Functions, $j_l(x)$, and , $n_l(x)$, the convention

$$j_0(x) = \frac{\sin x}{x} \quad , \quad n_0(x) = \frac{\cos x}{x} \quad \text{--------------(24)}$$

And for higher, l, the spherical Bessel Functions are determined from the recursion relation .

$$f_{l-1}(x) + f_{l+1}(x) = \frac{2l+1}{x} f_l(x) \quad \text{--------------------(25)}$$

where , $f_l(x)$, stands for , j_l , or , n_l .

Also, j_l' , & , n_l' , are determined from the relation ,

$$x\,f_l' = x\,f_l - x\,f_{l+1} \quad \text{----------(26)}$$

Consequently eq.(18) is written as :-

$$g_l(\varepsilon) = \sum_{\varepsilon_{nl}\le 0} 2(2l+1)\frac{1}{\sqrt{\pi}\,\Gamma}\ exp\left\{-\frac{1}{\Gamma^2}\left[\varepsilon - \left(N+\frac{3}{2}\right)\hbar\omega\right]^2\right\}$$

$$+\frac{2}{\pi}\frac{d\Delta_0(\varepsilon)}{d\varepsilon} + \frac{2}{\pi}(2l+1)\frac{d\Delta_l(\varepsilon)}{d\varepsilon} \quad \text{-----------------(27)}$$

where the first term of eq.(27) represent the partial level density for the bound states , and the second and third terms , represent the partial level density for the unbound states .

Also we have employed in this work the well – defined Strutnisky smoothing procedure.

$$g_{st}(\varepsilon) = \int g(\varepsilon')\,\delta(\varepsilon - \varepsilon' : \Gamma)\,d\varepsilon' \quad \text{----------------(28)}$$

using the function

$$\delta(\varepsilon : \Gamma) = \frac{1}{\sqrt{\pi}\,\Gamma}\exp[-(\varepsilon/\Gamma)^2]\ \ L_M^{1/2}[(\varepsilon/\Gamma)^2] \quad \text{---------- (29)}$$

Where, $L_M^{1/2}\ [(\varepsilon/\Gamma)^2]$, represents Lagurre Polynomial account for curvature correction . In this work we neglect this correction because we use the simple Harmonic potential instead of Saxon-Wood potential [7,8]. In our calculations we use $\Gamma = 1.5\,\hbar\omega$, with, $\hbar\omega = \frac{45}{A^{1/3}} - \frac{25}{A^{2/3}}$ MeV , where A is the mass number .

Total Level Density

As it has been mentioned that eq.(27) represent the partial level density for one shell , either proton or neutron shell.

Now for calculating the total level density, we take the summation of partial wave, that analyzed for, $l = 0,\ to, l_{max}$.

$$g(\varepsilon) = \sum_{l=0}^{l_{max}} g_l^z(\varepsilon) + \sum_{l=0}^{l_{max}} g_l^n(\varepsilon) \qquad \text{------------(30)}$$

where , $g_l^z(\varepsilon)$, stands for proton shell , while , $g_l^n(\varepsilon)$, stands for neutron shell , and for reduction we multiply one term of eq.(30) by 2 to contain the both shells , because $(N = 2n + l - 2)$ for the two shells (protons & neutron) are the same for the most nuclei, however the difference for some nuclei does not exceed than one, then eq.(30) can be written as follows ,

$$g(\varepsilon) = 2\sum_{l=0}^{l_{max}} g_l(\varepsilon) \text{---------- (31)}$$

Results and Discussions

As shown from Fig.1. that the level density increases with A for energies below than 4.4 MeV , while the level densities decreases with A for energies higher, and this behavior is interpreted that in low energies the neutron will spend a relative long interval inside the nucleus, according uncertainty principle $\Delta E\ \Delta t \geq \hbar$, which causes the increase of level densities at low energies.

Fig. 2 represents the results of present work, which shows that the level densities increases l_{max} , while Fig.3 represents the single particle level density that had been done by [1, 9] who used Saxon- Wood potential, and both results were done for a super heavy nucleus A=274 .

Conclusions

We conclude that this work can be used to calculate level density up to 10 – 17 MeV , with total angular momentum not exceed (34) because the potential that used in this work is a Harmonic – Oscillator potential, while using Saxon – Wood potential leads us to maximum energy about 350 MeV, with total angular momentum not exceed (50) .

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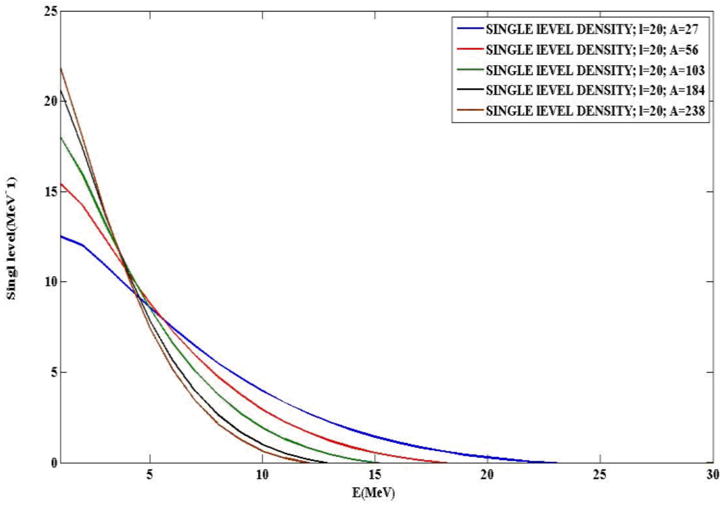

Figure (1): The calculated single level density for different nuclei.

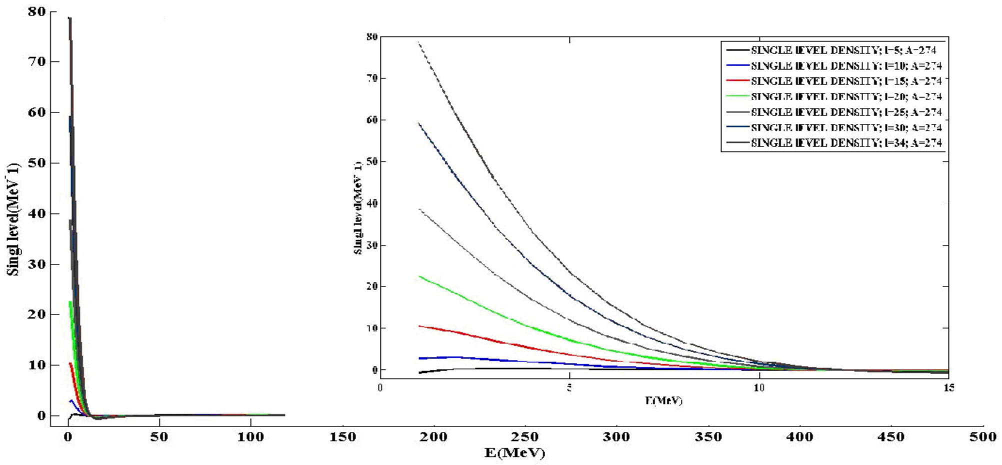

Figure (2) Single particle level density for A equal to 274 and different l_{max}