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Studying the Effect of Initial Conditions and System Parameters on the Behavior of a Chaotic Duffing System

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Abstract

This work presents a five-period chaotic system called the Duffing system, in which the effect of changing the initial conditions and system parameters d, g and w, on the behavior of the chaotic system, is studied. This work provides a complete analysis of system properties such as time series, attractors, and Fast Fourier Transformation Spectrum (FFT). The system shows periodic behavior when the initial conditions xi and yi equal 0.8 and 0, respectively, then the system becomes quasi-chaotic when the initial conditions xi and yi equal 0 and 0, and when the system parameters d, g and w equal 0.02, 8 and 0.09. Finally, the system exhibits hyperchaotic behavior at the first two conditions, 0 and 0, and the bandwidth of the chaotic signal becomes wider (2 a.u.) than in the first case. So, this system can be used in many physical applications, such as encrypting confidential information.

Article Info.

Keywords:
Chaos, Duffing System, Time Series, Attractor, nonlinear differential equation.

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1. Introduction

Chaos theory describes the qualities of the point at which stability moves to instability, or order moves to disorder. For example, unlike the behavior of a pendulum, which adheres to a predictable pattern, a chaotic system does not settle into a predictable pattern due to its nonlinear processes. Chaos can be defined as a dynamic, aperiodic system similar to noise but controlled by equations and sensitive to initial conditions [1]. These systems helped to understand and realize the behavior of systems that purify random behavior. Many chaotic models have been studied and applied in many fields of science, like electronic devices (optocoupler networks using LEDs) [2], secure communication using Lorenz systems [3, 4], Rossler systems [5, 6], Rossler-Chua systems [7, 8], Lorenz-Chua systems [9, 10], image encryption based on computer-generated holograms [11], mathematics [12], weather [13], the biological field [13], as well as the study of fluid motion within an organism [14]. The first scientist to study and work in the field of chaotic systems was Lorenz in 1963 [15]. A group of researchers introduced new models to understand more about the behavior of this strange system, such as Chua and Rossler [16], which led to great and rapid progress in the science of chaos. The equations of chaotic systems can be represented by electronic circuits, as was done by the scientist Chua, who built the first simple electronic circuit that shows complex, chaotic behavior [17]. The Chua chaotic system is a simple electronic circuit used in different fields, some of which are communications. Accordingly, many implemented electronic circuits, such as RLC circuits, digital filters [18], energy storage circuits [17], capacitor circuits and electronic oscillators [19], exhibit complex, chaotic behavior [20]. New chaotic systems have emerged that differ from their predecessors, which have been used in many scientific fields [21-32]. Recently, the chaotic Duffing system has been introduced and studied for use in secure communications.

The aim of this work is to find ideal values for the initial conditions that make the behavior of the system excessively chaotic to obtain a broad spectral bandwidth that can be used in the field of secure communications.

2. Modelling

Engineer George Duffing (1861-1944) made early studies of nonlinear, chaotic dynamical systems and introduced the Duffing oscillator [33]. The Duffing oscillator is a second-order nonlinear differential equation used to model some oscillators of the driving and damping equations. The equation is given by [33]:

$$\dot{x} = y \quad (1)$$

$$\dot{y} = x - x^3 - d.y + g.\cos\left(w\frac{time}{2}pi\right) \quad (2)$$

where: the (unknown) function x=x(t) is the displacement at time t, $\dot{x}$ is the first derivative of x with respect to time. Duffing's equation is considered a nonlinear differential equation because it contains the term x^3. The Duffing equations have parameters that control the amount of damping (d), the amplitude of the periodic driving force (g) and the angular frequency of the periodic driving force (w), which are real numbers equal to 0.02, 8 and 0.5, respectively at which the system becomes excessively chaotic. The behavior of the chaotic system is changing by changing the values of these parameters, so it may become oscillating, double-frequency, or quasi-chaotic.

Many computer programs and numerical integration methods were used to solve nonlinear differential equations of all kinds. The Berkeley Madonna program was used in this work using the numerical integration of the type of Runge-Kutta 4, as shown in Fig. 1.

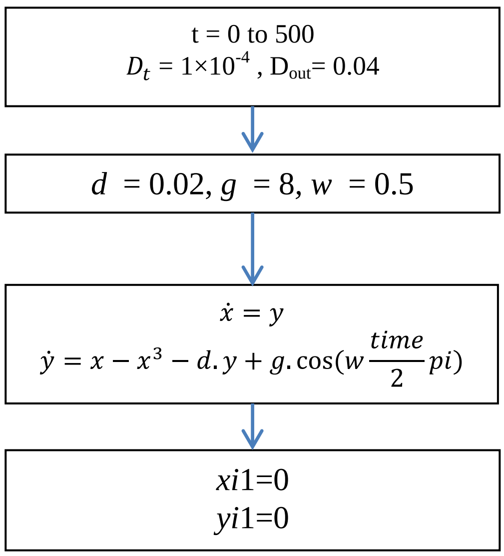

Figure 1: Flow-chart of Duffing model.

The computer program begins by determining the time period to study the behavior of the chaotic system (t), which is a changeable value according to the researcher's goal of work; it was determined to be 0 a.u. to 100 a.u. D_t is the interval time that equals 1×10^{-4} a.u., and D_{out} is the control time (which can be used to control how much output data is stored independently of the step size used in the computer), which equals 0.04 a.u. After defining the time period, the values of the parameters that make the Duffing system in a chaotic state are presented, which are d =0.02, g = 8, and w =0.5. Then the non-linear differential equations of the chaotic Duffing system are written in two dimensions, x and y. Finally, the values of the initial conditions xi1 and yi1, which make the behavior of the Duffing system in the chaotic state, are determined according to previous experiments on the mathematical model and are equal to 0 and 0, respectively.

The program takes time to calculate the mathematical operations and show the results, depending on the values of the initial conditions and the value of the interval time D_t. One of the benefits of the chaotic system is its use in secure communication applications [3-11].

3. Results and discussion

The effect of changing the initial conditions (xi, yi) on the behavior of the chaotic system was analyzed and studied. Two different conditions were taken (0.8, 0) and (0, 0). The MATLAB program, using Rang- Kuta 4th, was used to solve the differential equations, where the time scale taken was 500 a.u. The code results of Duffing system were as follows: The time series of all dynamics for Duffing system at initial condition (0.8, 0) are given in Figs. 2 and 3; Table 1 shows the amplitude values of all dynamics (peak to peak). From Figs. 2 and 3, it was noted that the system is in a periodic state evident through the fast Fourier transformation (FFT) spectra in x and y dynamics. Figs.4 and 5 show three distinctive peaks in the FFT spectrum of the x-dynamic and four distinctive peaks in the y-dynamic.

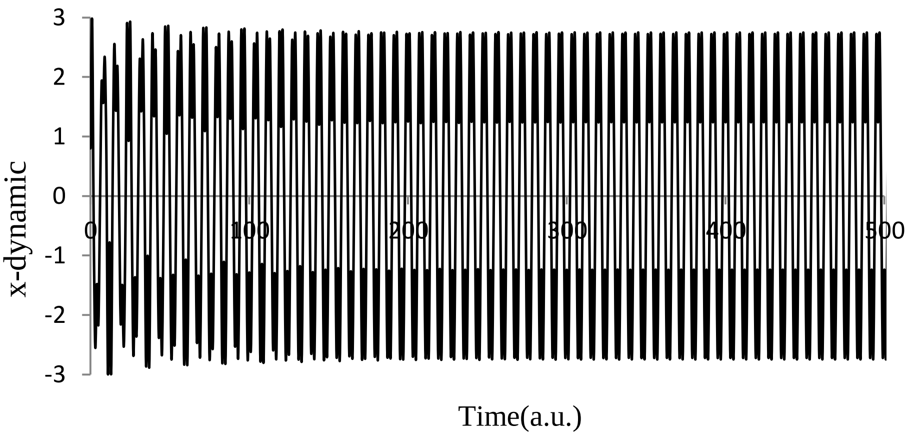

Figure 2: Time series in the x-dynamic. The system parameters d, g, and w equal 0.02, 8, and 0.5, respectively, at initial conditions xi and yi of 0.8 and 0, respectively.

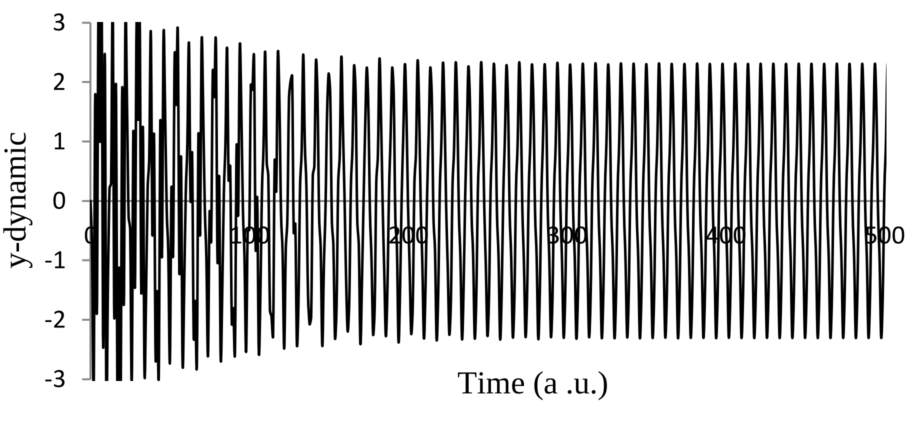

Figure 3: Time series in the y-dynamic. The system parameters d, g, and w equal 0.02, 8, and 0.5, respectively, at initial conditions xi and yi equal 0.8 and 0, respectively.

Table 1: The amplitude values of time series of the Duffing system

Parameters (d, g, and w)	**Initial conditions** x_i, y_i	$\mathbf{x_{p.p}}$**(a.u)**	$\mathbf{y_{p.p}}$**(a.u)**	**Dynamic state**
0.02, 8, and 0.5	0.8, 0	3:-3	3:-3	Periodic
0.02, 8, and 0.5	0, 0	3:-3.8	7:-7	Hyper-chaotic
0.02, 8 and 0.09	0, 0	3.5:-3.5	5.5:-5.5	Quasi-chaotic
0.02, 8 and 0.45	0, 0	3.75:-3.75	7:-7	Chaotic
0.02, 0.9 and 0.5	0, 0	2:-2	2.5:-2.5	Periodic
0.02, 0.2 and 0.5	0, 0	1.8:-1.8	1.55:-1.6	Chaotic

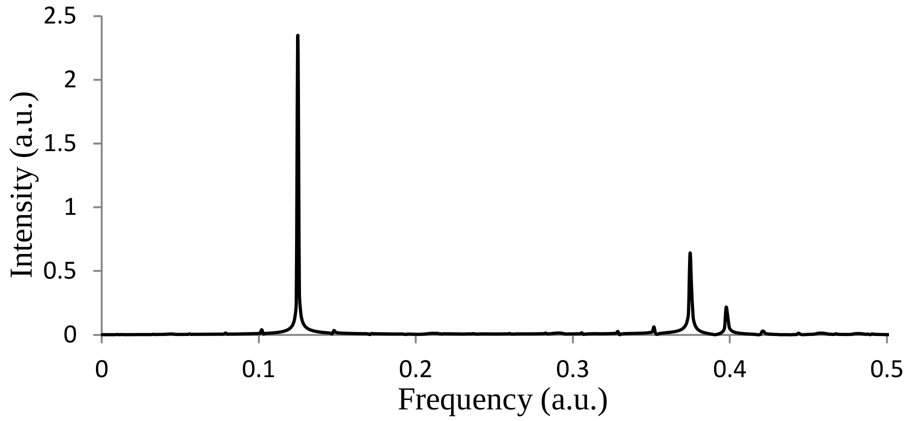

Figure 4: FFT spectrum in the x-dynamic. The system parameters d, g, and w equal 0.02, 8, and 0.5, respectively, at initial conditions xi and yi equal 0.8 and 0, respectively.

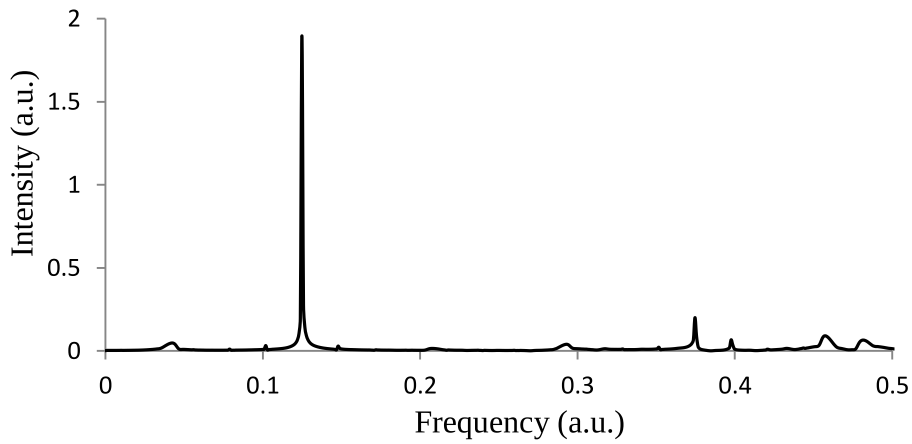

Figure 5: FFT spectrum inthe y-dynamic. The system parameters d, g, and w equal 0.02, 8, and 0.5, respectively, at initial conditions xi and yi equal 0.8 and 0, respectively.

Chaos theory describes the qualities of the point at which stability moves to instability or order moves to disorder. For example, unlike the behavior of a pendulum, which adheres to a predictable pattern, a chaotic system does not settle into a predictable pattern due to its nonlinear processes.

Fig. 6 represents the phase space in the (*y*-*x*) dynamic, with a high density in the phase space trajectories and approaches. Still, they do not match and do not intersect with some of them (note that trajectories never cross because the solution starting from any point in the plane is uniquely determined, so there cannot be two such solution curves starting at any given point). When a simple change is made in the value of initial conditions in the *x*-dynamic, hyper-chaotic behavior results, as seen in Fig. 7 and 8, which represent the time series in the *x* and *y*-dynamics, where the two dynamics have different behavior with each other in position and amplitude of its peaks. To analyze the bandwidth of Duffing system, Fig. 9 and 10 show the FFT spectra in the *x* and *y*-dynamics. It shows exponential decay behavior. The broadband and the values of maximum amplitudes, frequency bandwidth, and Full Width at Half Maximum (FWHM) are shown in Table 2. The reason for the increase in the bandwidth of the system is the emergence of many different frequencies with different amplitudes, which is one of the characteristics of the chaotic system.

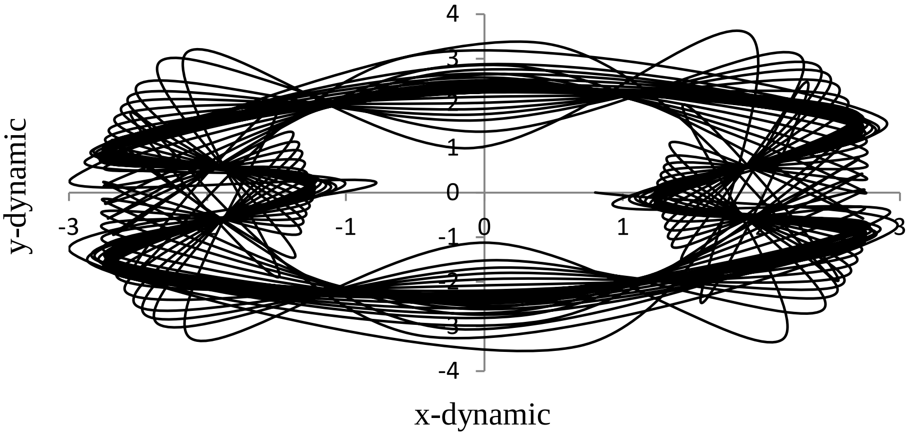

Figure 6: Phase space (y-x) attractor where the system parameters d, g, and w equal 0.02, 8, and 0.5, respectively, at initial conditions xi and yi equal 0.8 and 0, respectively.

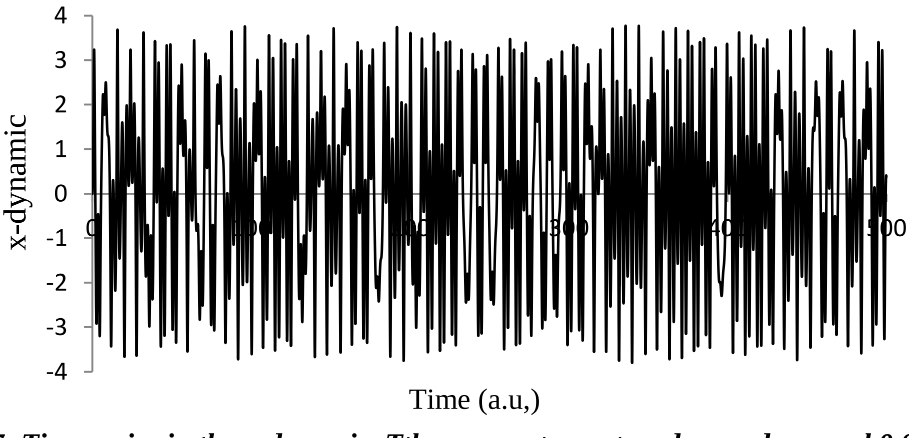

Figure 7: Time series in the x-dynamic. Tthe parameter system d, g, and w equal 0.02, 8 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

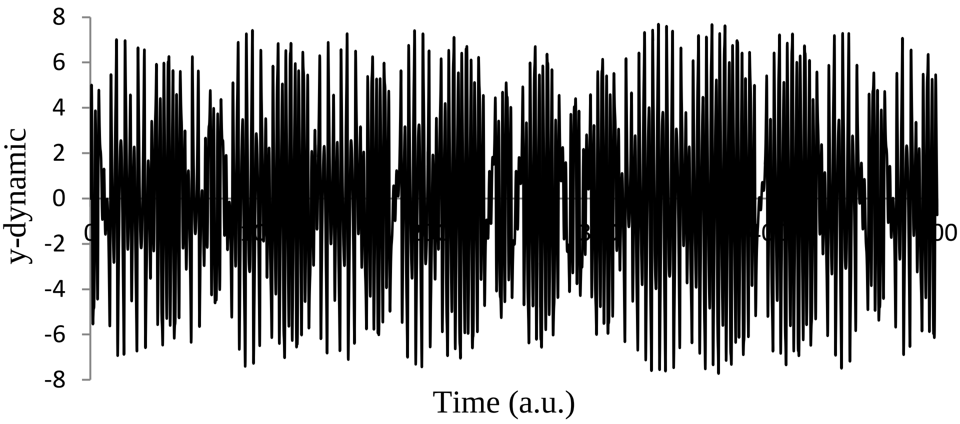

Figure 8: Time series in the y-dynamic. The system parameters d, g, and w equal 0.02, 8 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

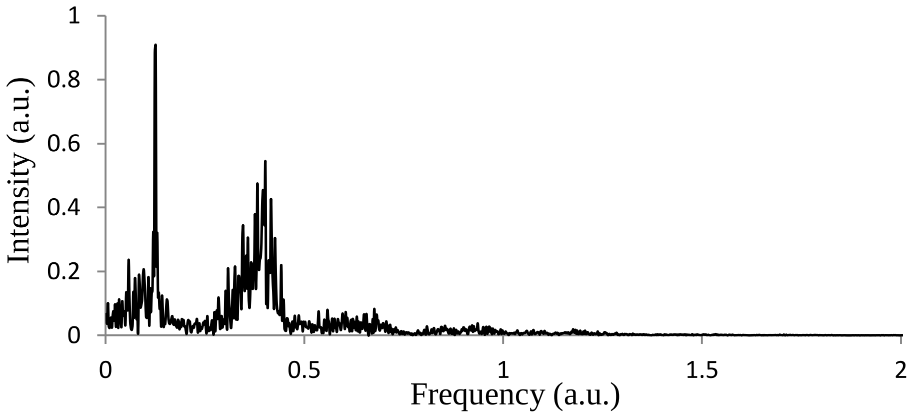

Figure 9: FFT spectrum in the x-dynamic. The system parameters d, g, and w equal 0.02,8 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

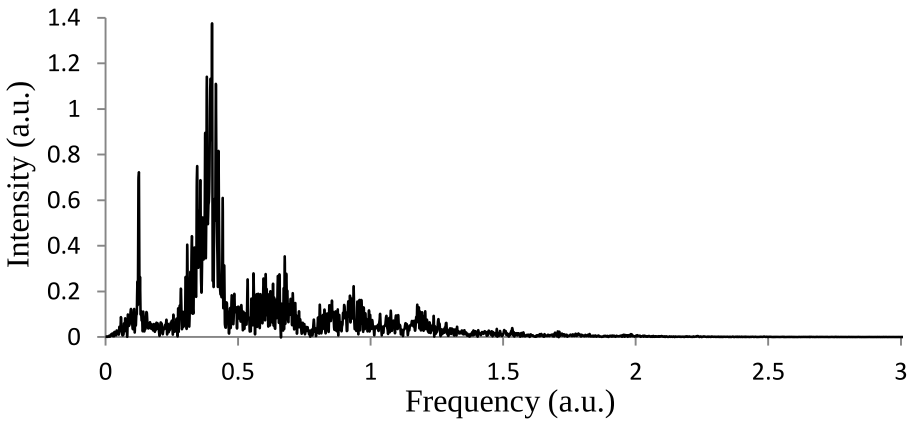

Figure 10: FFT spectrum in the y-dynamic. The system parameters d, g, and w equal 0.02 ,8 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

Table 2: FFT spectra parameters the Duffing system

Parameters (d, g, and w)	**Initial conditions x_i, y_i**	**dynamics**	**Amplitude (a.u)**	**Bandwidth (a.u)**
0.02, 8 , and 0.5	0.8, 0	*x*	0.95	1.25
0.02, 8 , and 0.5	0, 0	*y*	1.4	2
0.02, 8 and 0.09	0, 0	*x* and *y*	2 and 0.6	0.7 and 1.5
0.02, 8 and 0.45	0,0	*x* and *y*	1.6 and 1.3	1.2 and 1.5
0.02, 0.9 and 0.5	0,0	*x* and *y*	1 and 0.8	0.8 and 0.9
0.02, 0.2 and 0.5	0,0	*x* and *y*	0.45 and 0.45	0.6 and 1

The Lyapunov exponent can quantitatively reflect the chaotic performance of a system [34]. A hyper-chaotic system can be described as containing more than one positive Lyapunov exponent, which indicates that these systems expand in many directions, leading to the emergence of a very complex attractor. The maximal Lyapunov exponent can be defined as follows:

$$\lambda = \lim_{t\to\infty} . \lim_{|\delta z_\circ|} \frac{1}{t} \ln \frac{|\delta z\,(t)|}{|\delta z_\circ|} \qquad (3)$$

The limit $|\delta z_\circ| \to 0$ ensures the validity of the linear approximation at any time. For a discrete time, a system (maps or fixed-point iterations) $x_{n+1} = f(x_n) \quad x_0 + 1 =$ for an orbit starting with $x_\circ$ this translates into:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f(x_i)| \qquad (4)$$

The strange attractor for the two dynamics *x* and y is given in Fig. 11. Its shape has double scrolls with a complex topological structure. The double-scroll system is often described as having three nonlinear ordinary differential equations and a 3-segment piecewise-linear equation (see Chua's equations). This makes the system easily simulated numerically and easily manifested physically due to Chua's circuits' simple design.

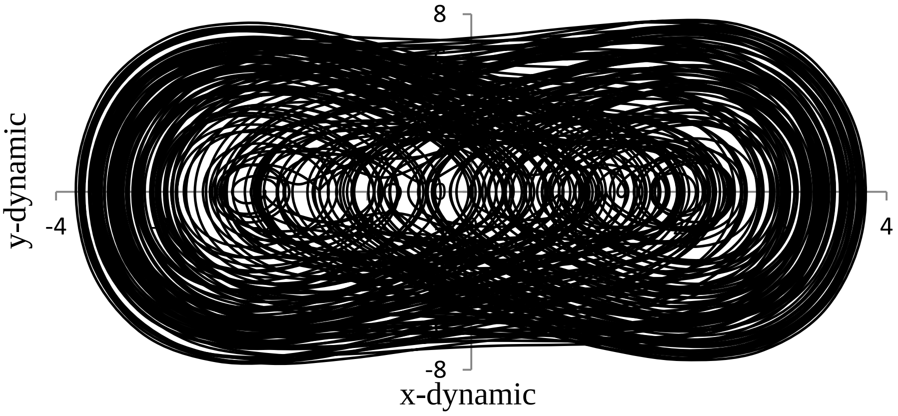

Figure 11: Phase space, (y-x) attractor. The system parameters d, g, and w equal 0.02, 8 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

One of the important tools for studying chaotic behavior is the effect of w and g system parameters on changing chaotic behavior. Two different values of the w parameter, 0.09 and 0.45, were taken in this work. Figs. 12 and 13 represent the time series in the x and y-dynamic, where this system manifested a quasi-chaotic behavior at this value; this is evident in the time series of these figures. This behavior appears in the FFT spectra and attractor, as shown in Figs. 14, 15, and 16. When w equals 0.45, the system converts to chaotic behavior, as shown in Figs.17-21. More details of this change are shown in Tables 1 and 2. In the same manner, the effect of the g value on the system's behavior was studied. When g equals 0.9, the system appears in a periodic state, as shown in Figs. 22-26: when g equals 0.2; the system becomes chaotic, as shown in Figs. 27-31.

Through what has been studied, it has been observed that the behavior of the Duffing system depends mainly on a set of initial conditions, which are responsible for achieving the chaotic state of the system. Many initial conditions were changed, and it was noted that the Duffing system converts from one case to another depending on the changes in the values of its initial conditions.

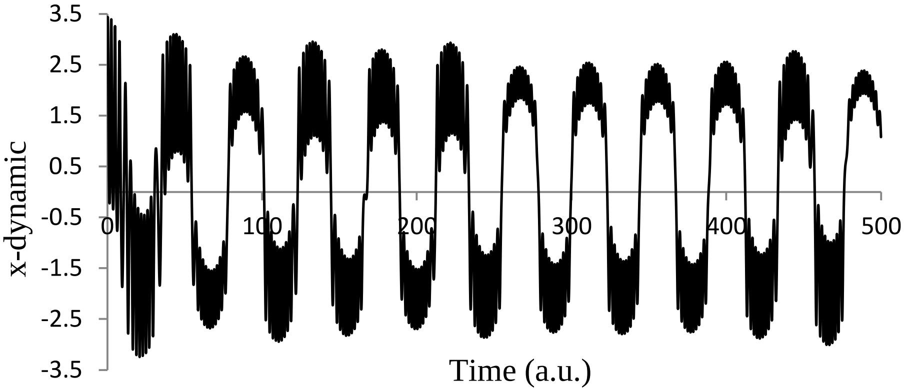

Figure 12: Time series in the x-dynamic. The system parameters d, g, and w equal 0.02, 8 and 0.09, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

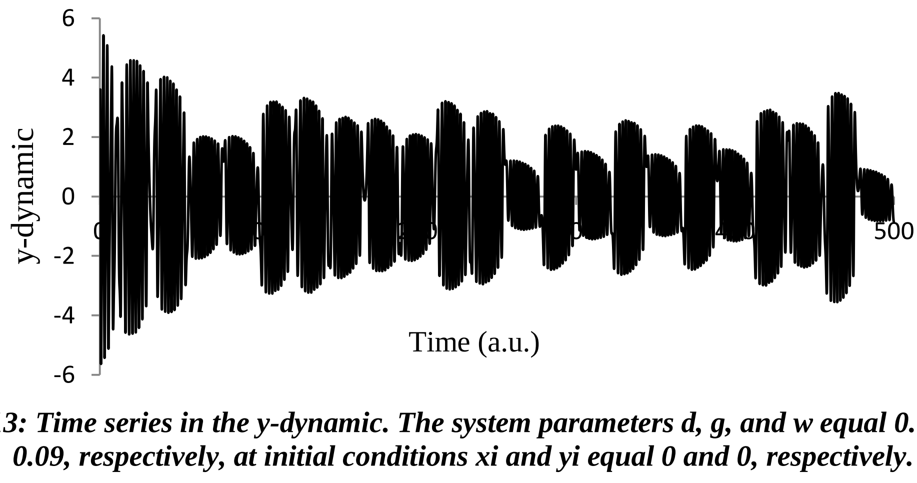

Figure 13: Time series in the y-dynamic. The system parameters d, g, and w equal 0.02, 8 and 0.09, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

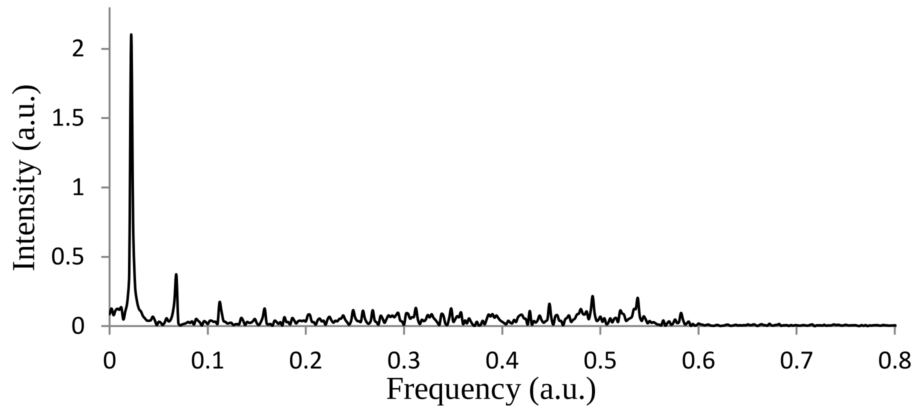

Figure 14: FFT spectrum in the x-dynamic. The system parameters d, g, and w equal 0.02, 8 and 0.09, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

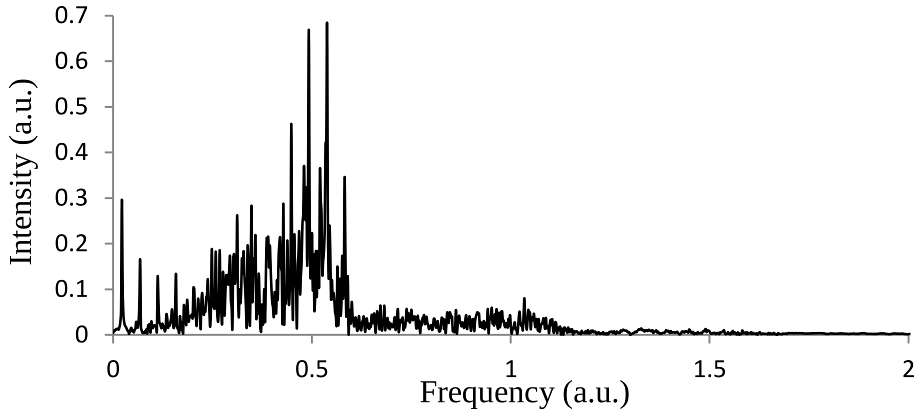

Figure 15: FFT spectrum in the y-dynamic. The system parameters d, g, and w equal 0.02, 8 and 0.09, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

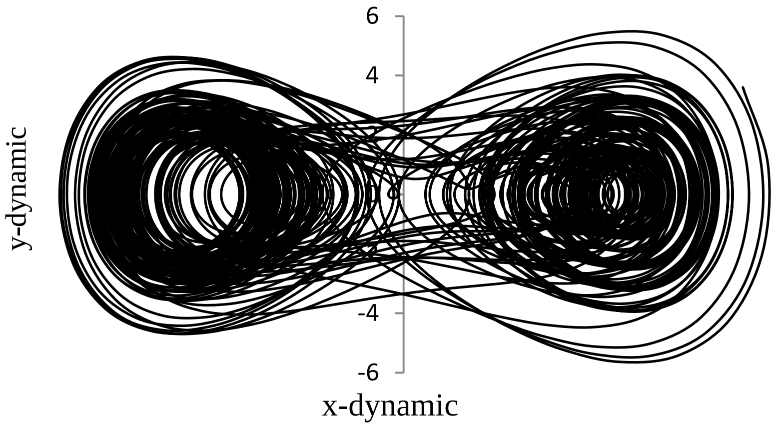

Figure 16: Phase space (y-x) attractor. The system parameters d, g, and w equal 0.02, 8 and 0.09, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

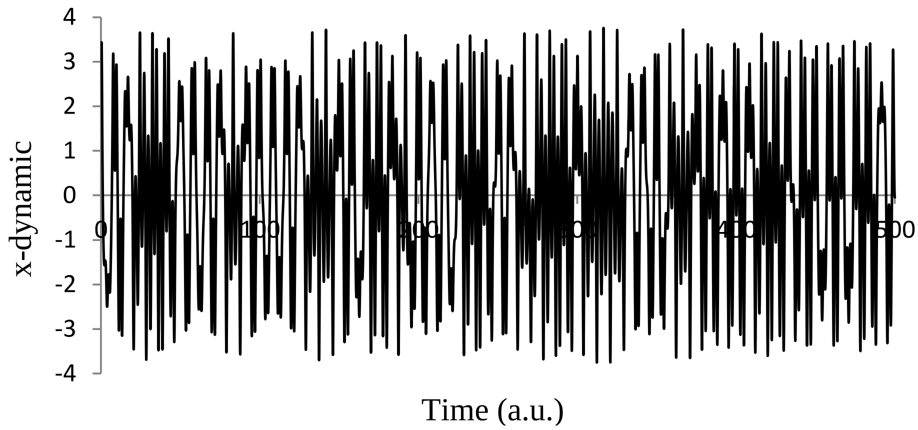

Figure 17: Time series in the x-dynamic. The system parameters d, g, and w equal 0.02, 8 and 0.45, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

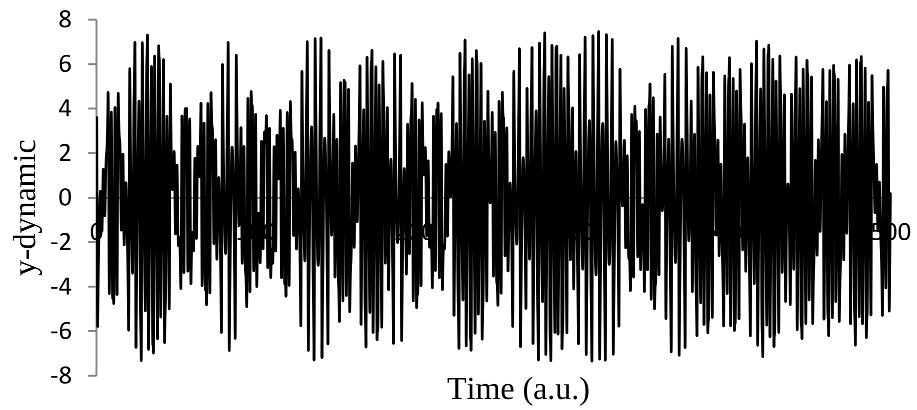

Figure 18: Time series in the y-dynamic. The system parameters d, g, and w equal 0.02, 8 and 0.45, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

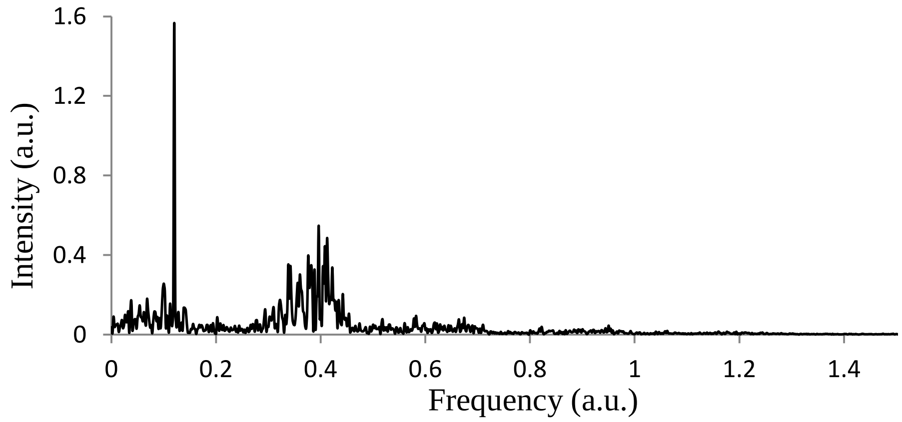

Figure 19: FFT spectrum in the x-dynamic. The system parameters d, g, and w equal 0.02, 8 and 0.45, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

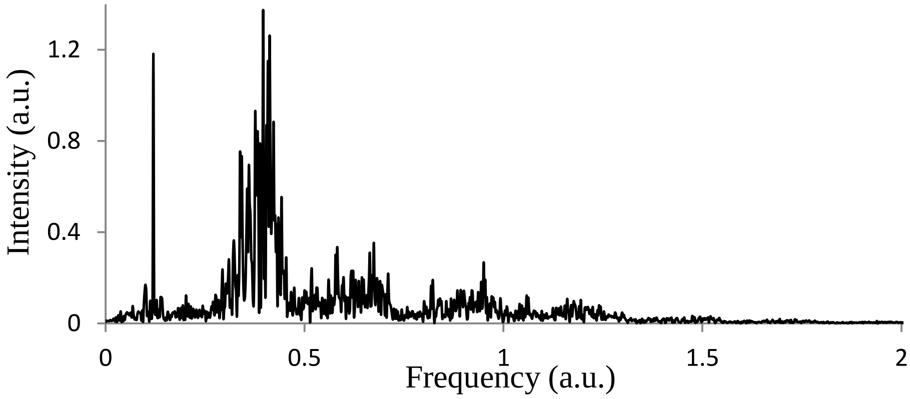

Figure 20: FFT spectrum in the y-dynamic. The system parameters d, g, and w equal 0.02, 8 and 0.45, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

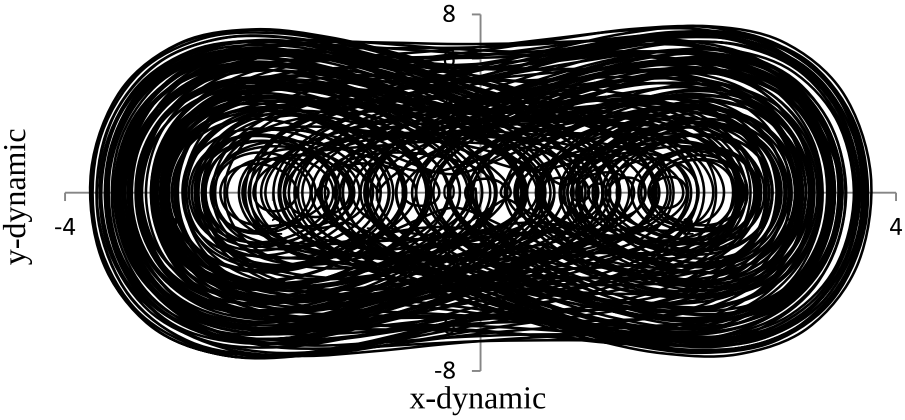

Figure 21: Phase space (y-x) attractor. The system parameters d, g, and w equal 0.02, 8 and 0.045, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

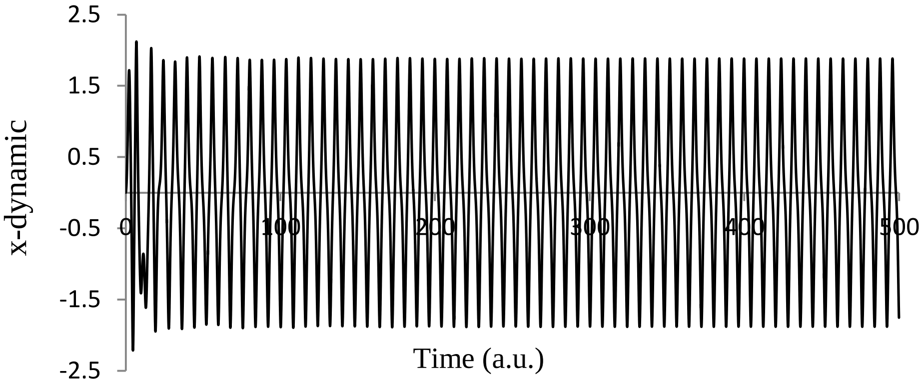

Figure 22: Time series in the x- dynamic. The system parameters d, g, and w equal 0.02, 0.9 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

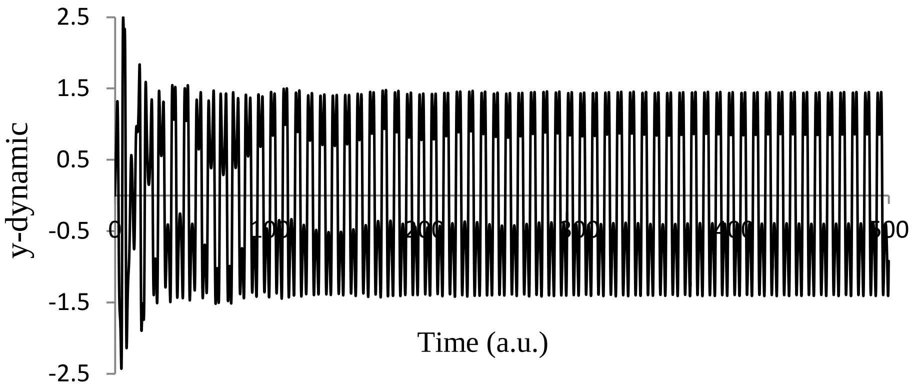

Figure 23: Time series in the y-dynamic. The system parameters d, g, and w equal 0.02, 0.9 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

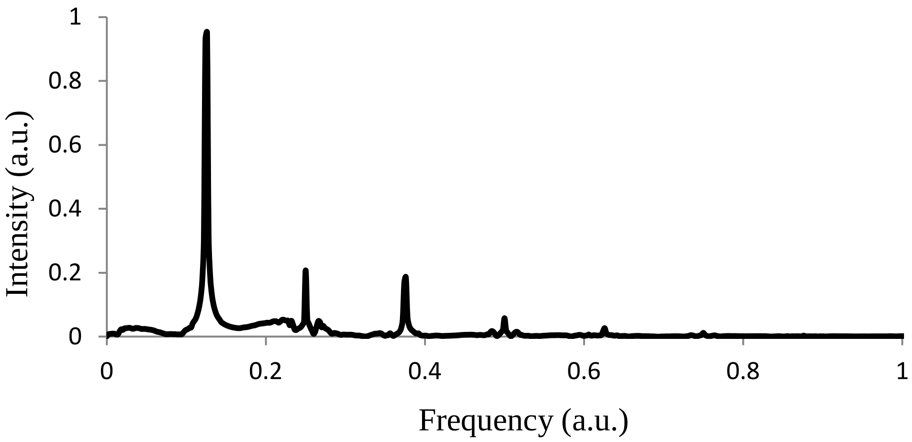

Figure 24: FFT spectrum in the x- dynamic. The system parameters d, g, and w equal 0.02, 0.9 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

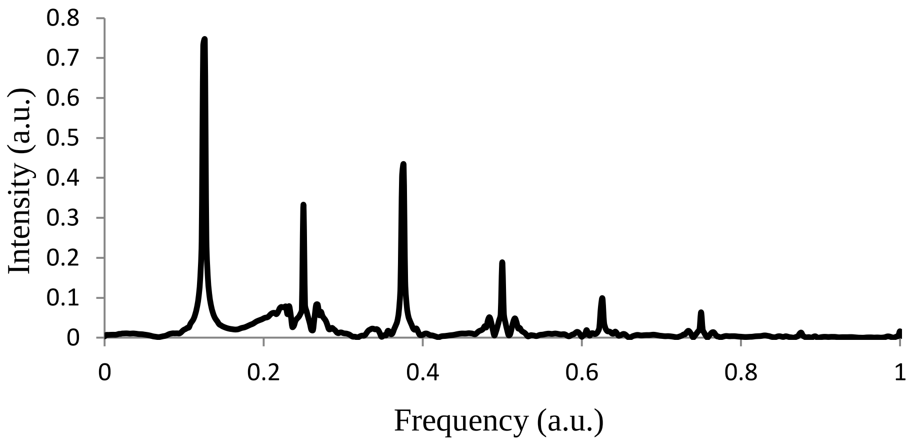

Figure 25: FFT spectrum in the y-dynamic. The system parameters d, g, and w equal 0.02, 0.9 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

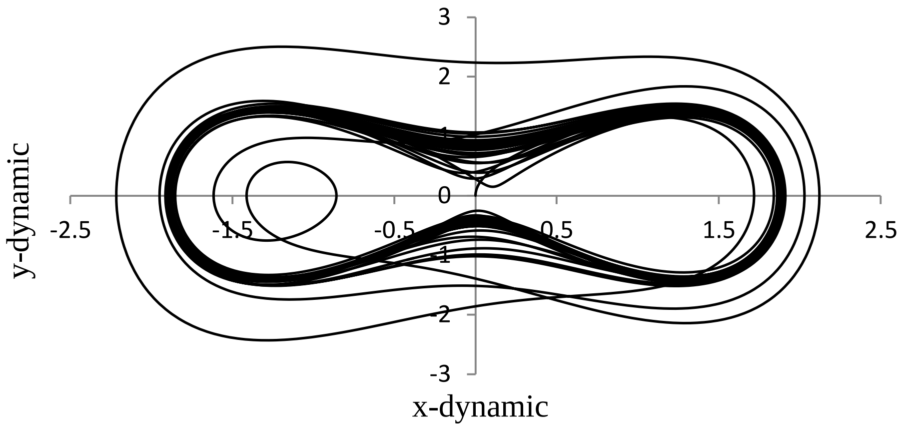

Figure 26: Phase space(y-x) attracter. The system parameters d, g, and w equal 0.02, 0.9 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

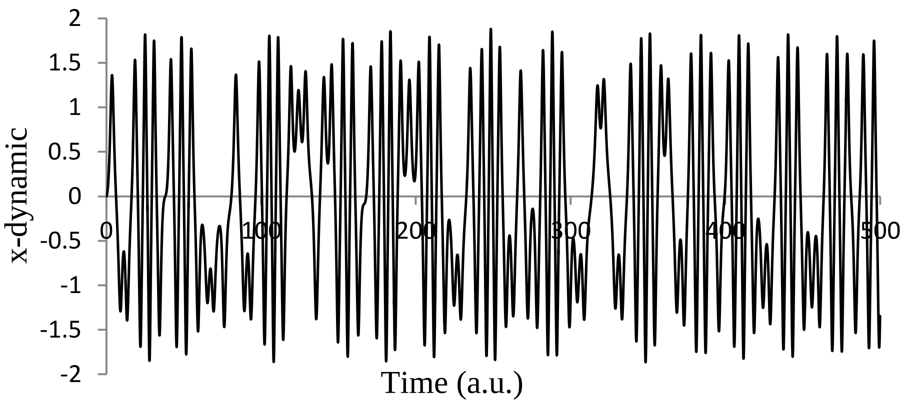

Figure 27: Time series in the x- dynamic. The system parameters d, g, and w equal 0.02, 0.2 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

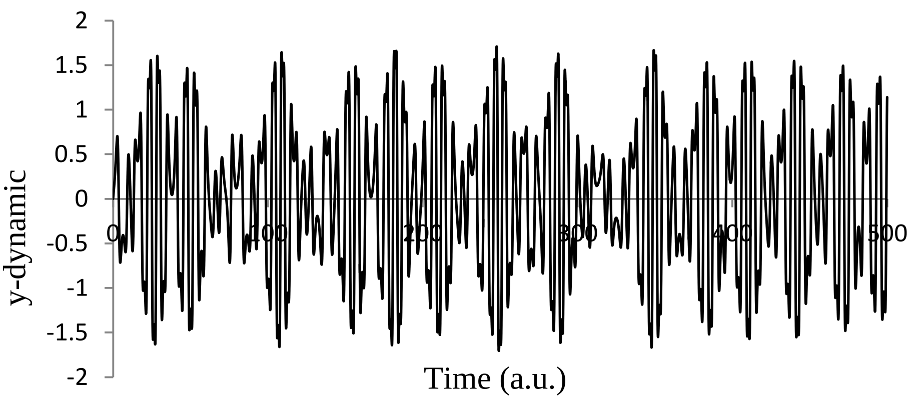

Figure 28: Time series in the y-dynamic. The system parameters d, g, and w equal 0.02, 0.2 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

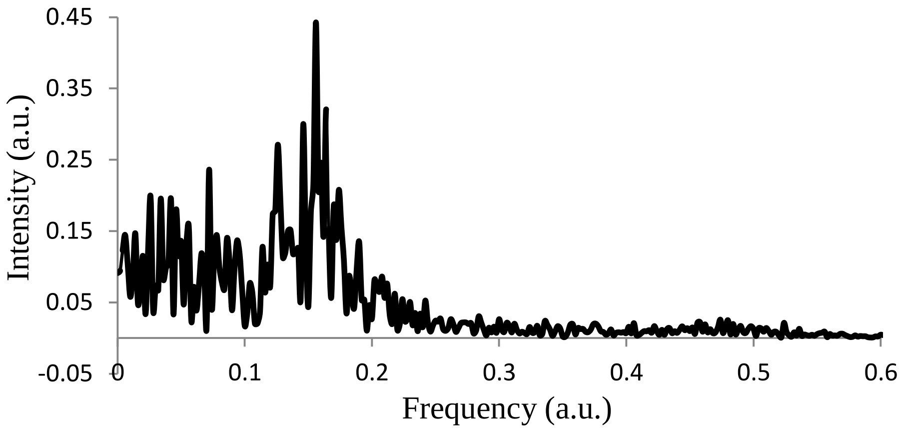

Figure 29: FFT spectrum in the x-dynamic. The system parameters d, g, and w equal 0.02, 0.2 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

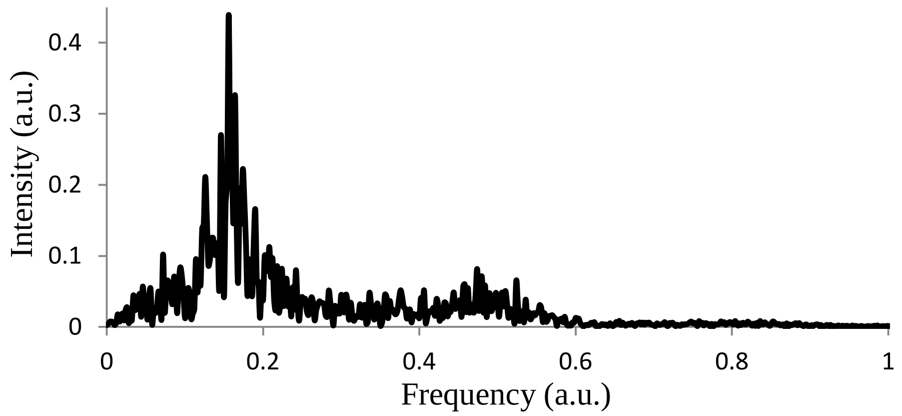

Figure 30: FFT spectrum in the y-dynamic. The system parameters d, g, and w equal 0.02, 0.2 and 0.5, respectively at initial conditions xi and yi equal 0 and 0, respectively.

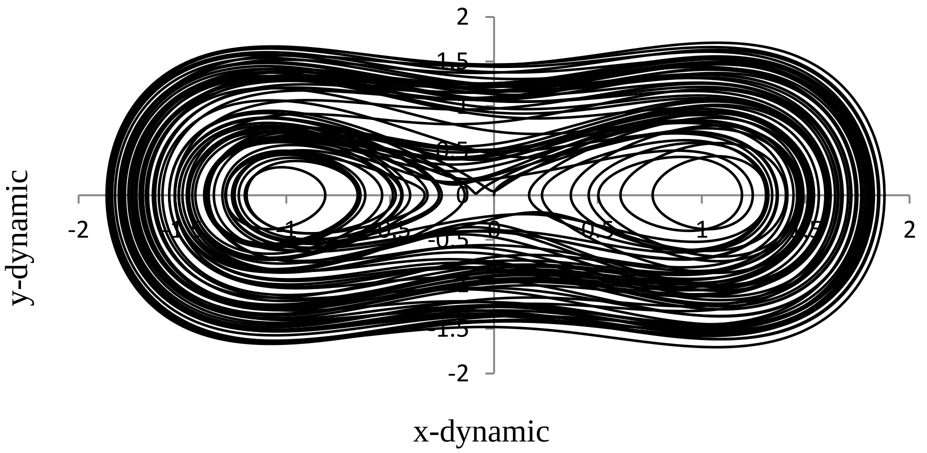

Figure 31: Phase space(y-x) attracter. The system parameters d, g, and w equal 0.02, 0.2 and 0.5, respectively, at initial conditions xi and yi equal 0 and 0, respectively.

4. Conclusions

This work concludes that the Duffing system depends entirely on a set of initial conditions (x_i, y_i) and also on the parameters of the system d, g, and w. The initial conditions (x_i, y_i) equal to 0.8, 0 makes the system periodic at parameters d, g, and w equal to 0.02, 8, and 0.5; the system becomes hyper chaotic at (x_i, y_i) of (0, 0) at the same system parameter values. This is what we are looking for in our future work on information coding or secure communications.

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Conflict of interest

Authors declare that they have no conflict of interest.

References

1. P. Kattan, Petra Books, 1 (2012).
2. K. M. Ibrahim and R. K. Jamal, Aust. J. Bas. Appl. Sci. **10**, 8 (2016).
3. R. K. Jamal and D. A. Kafi, Iraqi J. Phys. **14**, 51 (2016).
4. D. A. Kafi, R. K. Jamal, and K. A. Al-Naimee, Iraqi J. Phys. **14**, 51 (2016).
5. R. K. Jamal and D. A. Kafi, IOP Conference Series: Mater. Sci. Eng. (IOP Publishing, 2019). p. 012119.
6. R. K. Jamal and D. A. Kafi, Nonlin. Optic. Quant. Optic.: Con. Mod. Optic. **51**, 79 (2019).
7. R. K. Jamal, F. H. Ali, and F. A. Mutlak, Iraqi J. Sci. **62**, 2213 (2021).
8. R. S. Abdulaali, R. K. Jamal, and S. K. Mousa, Optic. Quant. Elect. **53**, 1 (2021).
9. D. A. Kafi, S. K. Mousa, and R. K. Jamal, Optic. Quant. Elect. **54**, 502 (2022).
10. S. K. Mousa and R. K. Jamal, Optic. Quant. Elect. **53**, 1 (2021).
11. I. A. Hamadi, R. K. Jamal, and S. K. Mousa, Optic. Quant. Elect. **54**, 33 (2022).
12. N. Hiroyuki, *Introduction to Chaos*. (Bristol, Philadelphia, IOP Publishing, 1999).
13. L. F. Olsen and H. Degn, Quart. Rev. Biophys. **18**, 165 (1985).
14. E. R. Weibel, Americ. J. Physiology-Lung Cell. Molec. Physio. **261**, L361 (1991).
15. E. N. Lorenz, J. Atmosph. Sci. **20**, 130 (1963).
16. M. Lakshmanan and K. Murali, *Chaos in Nonlinear Oscillators: Controlling and Synchronization*. Vol. 13. (USA, World scientific, 1996).
17. L. O. Chua and T. Lin, Inter. J. Cir. Theo. Appl. **18**, 541 (1990).
18. C. K. Tse, IEEE Trans. Cir. Syst. I: Fund. Theo. Appl. **41**, 16 (1994).
19. G. Poddar, K. Chakrabarty, and S. Banerjee, Elect. lett. **31**, 841 (1995).
20. M. J. Ogorzalek, IEEE Trans. Cir. Syst. **36**, 1221 (1989).
21. H. Kawakami, IEEE Trans. Cir. Syst. **31**, 248 (1984).
22. C. Li, I. Pehlivan, J. C. Sprott, and A. Akgul, IEICE Elect. Exp. **12**, 20141116 (2015).
23. A. Akgul, I. Moroz, I. Pehlivan, and S. Vaidyanathan, Optik **127**, 5491 (2016).
24. Z. Hou, N. Kang, X. Kong, G. Chen, and G. Yan, Inter. J. Bifur. Chaos **20**, 557 (2010).
25. A. A. Abdallah and A. K. Farhan, Iraqi J. Sci. **36**, 324 (2022).
26. R. S. Abdulaali and R. K. Jamal, Iraqi J. Sci. **63**, 556 (2022).
27. M. K. Ibraheem and R. K. Jamal, Optic. Quant. Elect. **54**, 614 (2022).
28. N. M. Ali and R. K. Jamal, Optic. Quant. Elect. **54**, 641 (2022).
29. M. W. Kadhim, D. A. Kafi, E. A. Abed, and R. K. Jamal, J. Optic., 1 (2023).

30. N. H. Aljahdaly and M. A. Alharbi, J. Low Freq. Noise, Vibrat. Acti. Cont. **41**, 1454 (2022).
31. Y. O. El-Dib, Math. Comp. Simul. **194**, 552 (2022).
32. Y. O. El-Dib, Inter. J. Dynam. Cont. **10**, 1148 (2022).
33. J. M. Thompson and H. B. Stewart, *Non-Linear Dynamics and Chaos*. 2^{nd} Ed. (Chichester, John Wiley & Sons, 1986).
34. W. Van Drongelen, *Signal Processing for Neuroscientists*. 2^{nd} Ed. (UK. USA, Elesevier, Academic press, 2018).

دراسة تأثير الشروط الأولية ومعلمات النظام على سلوك نظام دافنك الفوضوي

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الخلاصة

يقدم هذا العمل نظامًا فوضويًا من خمس فترات يسمى نظام دافنك، حيث يتم دراسة تأثير تغيير الظروف الأولية ومعلمات النظام d و g و w على سلوك النظام الفوضوي. قدم العمل تحليلاً كاملاً لخصائص النظام مثل السلاسل الزمنية والجاذب وطيف تحويل فورييه السريع (FFT). يبدو أن النظام يُظهر سلوكًا دوريًا عندما تكون الظروف الأولية xi و y_i تساوي 0.8 و 0 على التوالي، ثم يصبح النظام فوضويًا عندما تكون الظروف الأولية x_i و y_i تساوي 0 و 0 وتكون معلمات النظام d و g و w يساوي 0.02 و 8 و 0.09. أخيرًا يُظهر النظام سلوكًا فوضويًا مفرطًا في الشرطين الأولين 0 و 0 ويصبح عرض النطاق الترددي للإشارة الفوضوية أوسع من الحالة الأولى ليكون 2 a.u.، لذلك يمكن استخدامه في العديد من التطبيقات الفيزيائية على سبيل المثال في التشفير المعلومات بشكل سري.

الكلمات المفتاحيه: فوضى، نظام دوفينغ، متسلسلة زمنية، جاذب، معادلات التفاضلية الغير خطية.